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Bounds on the heat kernel of the Schrödinger operator in a random electromagnetic field

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Abstract

We obtain lower and upper bounds on the heat kernel and Green functions of the Schrödinger operator in a random Gaussian magnetic field and a fixed scalar potential. We apply the stochastic Feynman–Kac representation, diamagnetic upper bounds and the Jensen inequality for the lower bound. We show that if the covariance of the electromagnetic (vector) potential is increasing at large distances, then the lower bound decreases exponentially fast for large distances and a large time.

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1. Introduction

Random magnetic fields appear in many models of physical interest. In fundamental quantum theory the random magnetic field can be considered as a thermal (high temperature) part of the quantum electromagnetic field at finite temperature [1, 2]. The classical random magnetic field is discussed in optics [3] and in a description of an interaction of light with atoms [2]. A random magnetic field can arise as a Lagrange multiplier in models of interacting quantum particles creating four-fermion (or four-boson) interactions [4, 5]. In the Ginzburg–Landau theory of superconductivity [6] when fluctuations are taken into account then the electromagnetic field becomes a random Gaussian variable [7].

The effect of a random electric field in models of quantum mechanics has been well elaborated. Anderson localization [8] has been proved for a large class of models [9]. It seems that much less is known rigorously concerning the localization properties of the Hamiltonians with a random magnetic field (see [10] for a recent review; a special case of a magnetic field orthogonal to a plane and varying inside the plane is discussed in [11, 12]). Some aspects of localization (e.g., the integrated density of states [11, 12]) can be studied by means of the heat kernel of the Schrödinger Hamiltonian in a random electromagnetic field. The simplest estimate on the heat kernel follows from the diamagnetic inequality [13, 14] which bounds

the heat kernel in a magnetic field by that without the magnetic field. Such estimates are not interesting if we wish to detect the impact of the magnetic field on physical systems. Stronger upper bounds on the heat kernel in a (deterministic) magnetic field have been discussed in [15–19]. In these estimates the contribution of the lowest positive eigenvalues and eigenfunctions has been estimated.

We discuss a lower bound on the expectation value of the heat kernel. The heat kernel is gauge dependent. We explain which properties of the heat kernel do not depend on the choice of the gauge. Our result admits a fast decay of the heat kernel for large distances and a large time if the random vector potential has growing correlation functions. The well-known example of a constant magnetic field [19] shows that the exponential decay is really possible. The effect of the magnetic field upon the upper bounds is hard to derive. The classical Cwikel–Lieb–Rosenbljum bound on the number of eigenvalues [14] depends on momentum and the vector potential \mathbf{A} in the combination $|\mathbf{p} + \mathbf{A}|$. Hence, the dependence on the vector potential \mathbf{A} drops out. A more precise bound has been derived in [20, 21] which shows a dependence on the magnetic field. Results discussed in [11–15] display the discrete spectrum in the asymptotic behaviour of the heat kernel for a large time and large distances. Although our lower bound does not give an exact behaviour of the heat kernel, we nevertheless have a feeling that the exponentially decreasing lower bounds reflect an intrinsic property of the growing vector potential. We discuss in the last section a simple example of a non-trivial bound from above which supports our argument that growing vector potentials improve localization.

The heat kernel defines Green functions in Euclidean quantum field theory of an electromagnetic field interacting with a complex scalar field. The relation follows from Schwinger’s proper time formalism [22]: the correlation functions of the scalar field can be calculated from the heat kernel by an integral over the (proper) time. We consider the Ginzburg–Landau model with the Lagrangian

$$\mathcal{L} = \frac{1}{2} (i\hbar\nabla + \mathbf{A})\bar{\phi}(-i\hbar\nabla + \mathbf{A})\phi + \frac{1}{2}m^2|\phi|^2 + \beta|\phi|^4 + \frac{1}{4}F_{jk}F^{jk},$$

where $F_{jk} = \partial_j A_k - \partial_k A_j$.

The electromagnetic Lagrangian leads to ultraviolet problems which should be irrelevant for a large-distance behaviour. We regularize the electromagnetic potentials and discuss Euclidean scalar fields in a regular Gaussian electromagnetic field (the regularity can be achieved by an addition of higher-order derivatives of the electromagnetic field to the Lagrangian). We obtain lower and upper bounds on the correlation functions of the scalar field. Decay of correlation functions of the scalar field describes a disappearance of the long-range correlations of the order parameter in the Ginzburg–Landau model of superconductivity [6, 7].

2. The imaginary time evolution

We can apply simple inequalities only for the imaginary time evolution

$$\frac{d}{d\tau}\psi = \mathcal{A}\psi, \tag{1}$$

where $-\hbar\mathcal{A}(\mathbf{A}, V)$ is the Hamiltonian with a random vector potential \mathbf{A} and a scalar potential V (the scalar potential could have been random but its eventual randomness would not change our results concerning random magnetic fields, so we treat it as deterministic),

$$\hbar\mathcal{A} = -\frac{1}{2m}(-i\hbar\nabla + \mathbf{A})^2 - V. \tag{2}$$

The imaginary time can be treated as the inverse temperature β of the quantum Gibbs distribution

$$\hbar\beta = \tau.$$

In this section we assume that the scalar potential V is continuous and bounded from below and the sample functions of the vector potential \mathbf{A} are continuously differentiable (almost surely). In the next section we relax the assumptions on \mathbf{A} (admitting continuous transverse vector potentials).

Under our assumptions on the potentials for almost every sample path \mathbf{A} the solution of equation (1) defines a random semigroup T_τ [23] expressed in the form [14, 24]

$$\begin{aligned} \psi_\tau(\mathbf{x}) &\equiv (T_\tau \psi)(\mathbf{x}) \\ &= E \left[\exp \left(\frac{i}{\hbar} \int_0^\tau \mathbf{A}(\mathbf{x} + \sigma \mathbf{b}_s) \circ \sigma \, d\mathbf{b}_s - \frac{1}{\hbar} \int_0^\tau V(\mathbf{x} + \sigma \mathbf{b}_s) \, ds \right) \psi(\mathbf{x} + \sigma \mathbf{b}_\tau) \right], \end{aligned} \tag{3}$$

where

$$\sigma = \sqrt{\frac{\hbar}{m}} \tag{4}$$

and \mathbf{b} is the Brownian motion defined as the Gaussian process with the covariance

$$E[b_j(s)b_k(s')] = \delta_{jk} \min(s, s') \tag{5}$$

The stochastic integral in equation (3) is defined in the Stratonovitch sense [29]. We have

$$\sigma \int \mathbf{A} \circ d\mathbf{b}(s) = \sigma \int \mathbf{A} \, d\mathbf{b}(s) + \int \frac{\sigma^2}{2} \nabla \mathbf{A} \, ds, \tag{6}$$

where the integral on the rhs is in the Ito sense. Hence, if the vector potential is in the transverse gauge then the Stratonovitch and Ito integrals coincide.

Equation (3) defines the kernel $K_\tau^{(\mathbf{A}, V)}(x, y)$ as an integrable function on $R^D \times R^D$. Under the assumptions that V is a continuous function bounded from below and the sample paths of \mathbf{A} are continuously differentiable there exists a unique kernel continuous on $R^D \times R^D$ and determined by the formula [14, 25] (for a minimal set of assumptions which allow us to derive the stochastic representation of the kernel see [26])

$$\begin{aligned} K_\tau^{(\mathbf{A}, V)}(\mathbf{x}', \mathbf{x}) &\equiv \exp(\tau \mathcal{A})(\mathbf{x}', \mathbf{x}) = (2\pi\tau\sigma^2)^{-\frac{D}{2}} \exp\left(-\frac{1}{2\tau\sigma^2}(\mathbf{x} - \mathbf{x}')^2\right) \\ &\times E \left[\exp \left(\frac{i}{\hbar} \int_0^\tau \mathbf{A}(\mathbf{q}(s)) \circ d\mathbf{q}_s - \frac{1}{\hbar} \int_0^\tau V(\mathbf{q}_s) \, ds \right) \right]. \end{aligned} \tag{7}$$

Here

$$\mathbf{q}_s = \mathbf{x} + (\mathbf{x}' - \mathbf{x}) \frac{s}{\tau} + \sigma \sqrt{\tau} \mathbf{a} \left(\frac{s}{\tau} \right), \tag{8}$$

where the Brownian bridge \mathbf{a} is the Gaussian process on $[0, 1]$ starting in $\mathbf{0}$ at $s = 0$ and ending in $\mathbf{0}$ at $s = 1$ with the covariance

$$E[a_j(s)a_k(s')] = \delta_{jk}s'(1-s)$$

if $s' \leq s$. It can be expressed by the Brownian motion

$$\mathbf{a}(s) = (1-s)\mathbf{b} \left(\frac{s}{1-s} \right).$$

From the representation (7) there follows the diamagnetic upper bound (consequences of the stochastic representation are discussed in [13, 14]; for an analytic proof see [28])

Proposition 1 (Kato inequality). *Assume that \mathbf{A} is a continuously differentiable function and V is a continuous function bounded from below; then the version of the kernel which is continuous on $R^D \times R^D$ satisfies the inequality*

$$|K_\tau^{(\mathbf{A},V)}(\mathbf{x}, \mathbf{y})| \leq K_\tau^{(0,V)}(\mathbf{x}, \mathbf{y}) \tag{9}$$

for any $\tau, \mathbf{x}, \mathbf{y}$.

Let us note that if

$$\mathbf{A}' = \mathbf{A} + \nabla\chi, \tag{10}$$

then

$$\exp(\tau\mathcal{A}')(\mathbf{x}', \mathbf{x}) = \exp\left(\frac{i}{\hbar}\chi(\mathbf{x}') - \frac{i}{\hbar}\chi(\mathbf{x})\right) \exp(\tau\mathcal{A})(\mathbf{x}', \mathbf{x}). \tag{11}$$

The kernel is not invariant under the gauge transformations. However,

$$(\psi'_1, \exp(\tau\mathcal{A}')\psi'_2) = (\psi_1, \exp(\tau\mathcal{A})\psi_2) \tag{12}$$

if

$$\psi'_j = \exp\left(\frac{i}{\hbar}\chi\right) \psi_j. \tag{13}$$

We consider a random Gaussian electromagnetic field \mathbf{A}' with the mean equal to zero. We define the covariance G' of \mathbf{A}' in an arbitrary gauge as the expectation value

$$G'_{jk}(\mathbf{x}, \mathbf{x}') = \langle A'_j(\mathbf{x})A'_k(\mathbf{x}') \rangle. \tag{14}$$

It will be convenient to fix the gauge of the random vector potential and subsequently discuss a dependence of our results on the gauge. From equation (6) it can be seen that a transverse (Landau) gauge $\text{div } \mathbf{A} = 0$ is distinguished from the point of view of the path integral (if $\text{div } \mathbf{A} = 0$ in a distributional sense, then a continuity of \mathbf{A} is sufficient for the stochastic representation (7) of the kernel). If we wish to calculate the heat kernel in another gauge \mathbf{A}' , then we need the gauge transformation with

$$\chi = \Delta^{-1} \text{div } \mathbf{A}', \tag{15}$$

transforming \mathbf{A}' to the transverse gauge. The covariance G of \mathbf{A} is related to that of \mathbf{A}' by the formula

$$G_{jk}(\mathbf{x}, \mathbf{x}') = (\delta_{jr} - \partial_j \partial_r \Delta^{-1})(\delta_{km} - \partial'_k \partial'_m \Delta'^{-1})G'_{rm}(\mathbf{x}, \mathbf{x}'). \tag{16}$$

We calculate the Gaussian expectation value of the kernel (7) using the definition of the Gaussian variable

$$\langle \exp(i\langle \mathbf{A}, \mathbf{f} \rangle) \rangle = \exp\left(-\frac{1}{2}\langle (\mathbf{A}, \mathbf{f})^2 \rangle\right). \tag{17}$$

From this formula we can see that a change of the gauge involves the factor

$$\exp\left(-\frac{1}{2\hbar^2}\left\langle \left(\chi(\mathbf{x}) - \chi(\mathbf{x}') + \int \mathbf{A}' \circ d\mathbf{q}\right)^2 \right\rangle\right) = \exp\left(-\frac{1}{2\hbar^2}\left\langle \left(\int \mathbf{A} d\mathbf{q}\right)^2 \right\rangle\right) \tag{18}$$

in the kernel (7).

3. Bounds on the heat kernel and Green functions

We can calculate the expectation value of the kernel over the magnetic field

$$\begin{aligned} \langle K_\tau^{(\mathbf{A}, V)}(\mathbf{x}', \mathbf{x}) \rangle &\equiv \langle \exp(\tau \mathcal{A}(\mathbf{A}, V))(\mathbf{x}', \mathbf{x}) \rangle = (2\pi\tau\sigma^2)^{-\frac{D}{2}} \exp\left(-\frac{1}{2\tau\sigma^2}(\mathbf{x} - \mathbf{x}')^2\right) \\ &\times E \left[\exp\left(-\frac{1}{2\hbar^2} \left\langle \left(\int_0^\tau \mathbf{A}(\mathbf{q}(s)) \circ d\mathbf{q}_s \right)^2 \right\rangle - \frac{1}{\hbar} \int_0^\tau V(\mathbf{q}_s) ds \right) \right]. \end{aligned} \tag{19}$$

In formula (19) the Stratonovitch integral can be expressed by the Ito integral (the integrals coincide for transverse vector fields). It follows from equation (19) that

$$\langle K_\tau^{(\mathbf{A}, V)}(\mathbf{x}', \mathbf{x}) \rangle \leq K_\tau^{(0, V)}(\mathbf{x}', \mathbf{x}),$$

which is already a consequence of the Kato inequality (9).

We rewrite the Ito integral in equation (19) in the form (we can prove this equality differentiating both sides of equation (20) by means of the Ito formula [29] and subsequently integrating it again, see also [30])

$$\left(\int_0^\tau \mathbf{A} d\mathbf{q} \right)^2 = 2 \int_0^\tau \mathbf{A} d\mathbf{q}_s \int_0^s \mathbf{A} d\mathbf{q}_{s'} + \int_0^\tau \mathbf{A}^2 \frac{ds}{\tau}. \tag{20}$$

Let us note that

$$\begin{aligned} \int_0^\tau \mathbf{A}(\mathbf{q}(s)) \circ d\mathbf{q}_s &= \int_0^\tau \mathbf{A}(\mathbf{q}(s)) d\mathbf{q}_s + \frac{\sigma}{2} \int_0^\tau \operatorname{div} \mathbf{A}(\mathbf{q}(s)) ds \\ &= \frac{\sigma}{2} \int_0^\tau \operatorname{div} \mathbf{A}(\mathbf{q}(s)) ds - \sigma\sqrt{\tau} \int_0^\tau \frac{ds}{\tau} \mathbf{A}(\mathbf{q}(s)) \mathbf{b} \left(\frac{s}{\tau-s} \right) \\ &\quad + \sigma\sqrt{\tau} \int_0^\tau \mathbf{A}(\mathbf{q}(s)) \left(1 - \frac{s}{\tau} \right) d\mathbf{b} \left(\frac{s}{\tau-s} \right) - \int_0^\tau \mathbf{A}(\mathbf{q}(s)) (\mathbf{x} - \mathbf{x}') \frac{ds}{\tau} \end{aligned} \tag{21}$$

consists of four terms which could behave in a different way for large distances.

We shall discuss either a random electromagnetic potential \mathbf{A} which is bounded (in a certain gauge) or a random electromagnetic potential which is scale invariant and growing with the distance

$$\mathbf{A}(\lambda\mathbf{x}) \simeq \lambda^\gamma \mathbf{A}(\mathbf{x}), \tag{22}$$

where the approximate equality means that the random fields on both sides have the same correlation functions. The scale invariance is assumed only for convenience. In general, we can treat the scale invariance (22) as an asymptotic behaviour for large distances.

When $\gamma > 0$ then a scale-invariant random field in the transverse gauge (16) must have the covariance (in the Feynman gauge we could interpret this vector field as D -independent Levy's D -dimensional Brownian sheets [31])

$$G_{jk}(\mathbf{x}, \mathbf{x}') = (\delta_{jr} - \partial_j \partial_r \Delta^{-1})(\delta_{kr} - \partial'_k \partial'_r \Delta'^{-1})(|\mathbf{x}|^{2\gamma} + |\mathbf{x}'|^{2\gamma} - |\mathbf{x} - \mathbf{x}'|^{2\gamma}), \tag{23}$$

with $\gamma < 1$. The derivatives on the rhs of equation (23) should be understood in the distributional sense. The sample paths of \mathbf{A} are Hölder continuous with an index α arbitrarily close to γ [27]. A random vector field \mathbf{A} which is only Hölder continuous does not satisfy the conditions for the representation of the kernel formulated in section 2. However, the kernel (7) for \mathbf{A} in the transverse gauge can be defined under the weaker condition of Hölder continuity [26]. The Ito stochastic integral in equation (7) is defined for any continuous \mathbf{A} . The expectation value (7) determines a jointly continuous function of $\tau, \mathbf{x}, \mathbf{y}$ for continuous

potentials \mathbf{A} and V (V bounded from below). Then, the kernel (7) defines a random semigroup T_τ [23] as a result of the Markov property of the Brownian motion and the multiplicative property of the exponentials [14]. The Hamiltonian $-\hbar\mathcal{A}$ (together with its domain) is defined as the generator of T_τ . Alternatively, we could introduce a covariance G_κ of the vector potential which is infinitely differentiable and depending on a regularizing parameter κ . We consider expectation values of the kernel with a regularized electromagnetic field. Subsequently, we take the limit $G_\kappa \rightarrow G$ of expectation values (where G is defined in equation (23)). The inequalities discussed in this paper are preserved under such limits.

We can obtain a lower bound on the heat kernel from the Jensen inequality [32, 33] as applied in equation (19) to an average over the Brownian motion

$$\langle \exp(\tau\mathcal{A})(\mathbf{x}', \mathbf{x}) \rangle \geq (2\pi\tau\sigma^2)^{-\frac{D}{2}} \exp\left(-\frac{1}{2\tau\sigma^2}(\mathbf{x} - \mathbf{x}')^2\right) \times \exp\left(-\frac{1}{2\hbar^2} E \left[\left\langle \left(\int_0^\tau \mathbf{A}(\mathbf{q}(s)) \circ d\mathbf{q}_s \right)^2 \right\rangle \right] - \frac{1}{\hbar} \int_0^\tau V(\mathbf{q}(s)) ds \right). \quad (24)$$

On the basis of equation (24) and the diamagnetic inequality (9) (proposition 1) we obtain the following bounds (the terms on the lhs of equations (25) and (28) can be related by some inequalities, but we keep this form of the inequalities in order to make the origin of these terms visible in such a form as they come from equation (24)).

Theorem 2. Assume that $c \leq V \leq a$ and the covariance of the electromagnetic field $\langle A_j(\mathbf{x})A_k(\mathbf{x}') \rangle$ in the transverse gauge is continuous and bounded. Then, there exists a constant $C > 0$ and positive constants a_j such that

$$C(2\pi\tau\sigma^2)^{-\frac{D}{2}} \exp\left(-\frac{1}{2\tau\sigma^2}(\mathbf{x} - \mathbf{x}')^2\right) \exp\left(-a_1\hbar^{-2}(\mathbf{x} - \mathbf{x}')^2 - a_2\hbar^{-\frac{3}{2}}|\mathbf{x} - \mathbf{x}'|\sqrt{\tau} - a\hbar^{-1}\tau\right) \leq \langle K_\tau(\mathbf{x}', \mathbf{x}) \rangle \leq \exp\left(-\frac{c}{\hbar}\tau\right) (2\pi\tau\sigma^2)^{-\frac{D}{2}} \exp\left(-\frac{1}{2\tau\sigma^2}(\mathbf{x} - \mathbf{x}')^2\right). \quad (25)$$

Theorem 3

(i) Assume that \mathbf{A} is transverse and continuous and V is continuous and bounded from below; then the kernel satisfies the upper bound

$$\langle K_\tau^{(\mathbf{A}, V)}(\mathbf{x}', \mathbf{x}) \rangle \leq (2\pi\tau\sigma^2)^{-\frac{D}{2}} \exp\left(-\frac{1}{2\tau\sigma^2}(\mathbf{x} - \mathbf{x}')^2\right) \int_0^\tau \frac{ds}{\tau} E \left[\exp\left(-\frac{\tau}{\hbar} V(\mathbf{q}_s)\right) \right] = (2\pi\sigma^2\tau)^{-\frac{D}{2}} \exp\left(-\frac{1}{2\tau}(\mathbf{x} - \mathbf{x}')^2\right) \int_0^1 ds \int d\mathbf{y} (2\pi)^{-\frac{D}{2}} \exp\left(-\frac{\mathbf{y}^2}{2}\right) \times \exp\left(-\frac{\tau}{\hbar} V(\mathbf{x} + (\mathbf{x}' - \mathbf{x})s + \sqrt{\tau}\sigma s(1-s)\mathbf{y})\right). \quad (26)$$

(ii) Assume that the vector potential in the transverse gauge is scale invariant and growing with the scale index γ (equation (23)); the scalar potential is a continuous function bounded from below and for certain $a > 0$ and $B > 0$

$$V(\mathbf{x}) \leq B|\mathbf{x}|^{2\beta} + a. \quad (27)$$

Then there exists a constant $C > 0$ and some positive constants a_j such that

$$\langle K_\tau(\mathbf{x}', \mathbf{x}) \rangle \geq C(2\pi\sigma^2\tau)^{-\frac{D}{2}} \exp\left(-\frac{1}{2\sigma^2\tau}(\mathbf{x} - \mathbf{x}')^2\right) \times \exp\left(-a_1\hbar^{-2}(|\mathbf{x}|^{2\gamma} + |\mathbf{x}'|^{2\gamma})(\mathbf{x} - \mathbf{x}')^2 - a_2\hbar^{-\frac{3}{2}}(|\mathbf{x}|^{2\gamma} + |\mathbf{x}'|^{2\gamma})|\mathbf{x} - \mathbf{x}'|\sqrt{\tau}\right)$$

$$\begin{aligned}
 & -a_3\hbar^{-1}(|\mathbf{x}|^{2\gamma} + |\mathbf{x}'|^{2\gamma})\tau - a_4\hbar^{-\frac{3}{2}+\gamma}\tau^{\frac{1}{2}+\gamma}|\mathbf{x} - \mathbf{x}'| - a_5\hbar^{-1+\gamma}\tau^{1+\gamma} \\
 & - a_6\hbar^{-2+\gamma}|\mathbf{x} - \mathbf{x}'|^{2\gamma} - a_7\hbar^{-1}(|\mathbf{x}|^{2\beta} + |\mathbf{x}'|^{2\beta})\tau - a_8\hbar^{-1+\beta}\tau^{1+\beta} - a\hbar^{-1}\tau.
 \end{aligned}
 \tag{28}$$

Remarks

1. The upper bound (26) does not depend on the vector potential as a consequence of the Kato inequality (9). The final inequality in equation (26) follows from the Jensen inequality as applied to the time integral.
2. By a scale transformation (for $V = 0$) we could obtain (an expectation value over a scale-invariant magnetic field (23))

$$\langle K_\tau^{(A,0)}(\mathbf{x}', \mathbf{x}) \rangle = (2\pi\sigma^2\tau)^{-\frac{D}{2}} \exp\left(-\frac{1}{2\sigma^2\tau}(\mathbf{x} - \mathbf{x}')^2\right) \exp\left(-\tau^{1+\gamma}F\left(\tau^{-\frac{1}{2}}\mathbf{x}, \tau^{-\frac{1}{2}}\mathbf{x}'\right)\right),
 \tag{29}$$

where the function F has to be determined by an explicit calculation. The lower bound (28) is in agreement with the scaling (29); it gives an upper bound for the function F .

The heat kernel will depend on the gauge. However, its diagonal

$$\begin{aligned}
 \langle \exp(\tau\mathcal{A})(\mathbf{x}, \mathbf{x}) \rangle & \geq C(2\pi\tau)^{-\frac{D}{2}} \exp(-2a_3\hbar^{-1}|\mathbf{x}|^{2\gamma}\tau - a_5\hbar^{-1+\gamma}\tau^{1+\gamma} \\
 & - 2a_7\hbar^{-1}|\mathbf{x}|^{2\beta}\tau - a_8\hbar^{-1+\beta}\tau^{1+\beta} - a\hbar^{-1}\tau)
 \end{aligned}
 \tag{30}$$

is gauge invariant. The diagonal of the heat kernel is equal to the Laplace transform of the integrated density of states [11, 25]. The integral over the diagonal

$$\langle \text{Tr}(\exp(\tau\mathcal{A})) \rangle = \left\langle \sum_n \exp(-\tau\epsilon_n(\mathbf{A}, V)) \right\rangle = \int d\mathbf{x} \langle \exp(\tau\mathcal{A})(\mathbf{x}, \mathbf{x}) \rangle
 \tag{31}$$

is expressing the sum over eigenvalues $\epsilon_n(\mathbf{A}, V)$ of the Hamiltonian $-\hbar\mathcal{A}$. We have

Corollary 4. For any $\tau > 0$ there exists a constant $C > 0$ such that

$$\begin{aligned}
 C\tau^{-\nu} \exp(-a_5\hbar^{-1+\gamma}\tau^{1+\gamma} - a_8\hbar^{-1+\beta}\tau^{1+\beta} - a\hbar^{-1}\tau) & \leq \langle \text{Tr}(\exp(\tau\mathcal{A})) \rangle \\
 & \leq (2\pi\tau\sigma^2)^{-\frac{D}{2}} \int d\mathbf{x} \exp\left(-\frac{\tau}{\hbar}V(\mathbf{x})\right).
 \end{aligned}
 \tag{32}$$

Here

$$\nu = \frac{D}{2} \left(1 + \frac{1}{\rho}\right),
 \tag{33}$$

with $\rho = \max(\gamma, \beta)$

Remarks

1. The factor $\tau^{-\nu}$ in the lower bound on the lhs of equation (32) is non-trivial only for a small time; for a large time the exponential terms decay much faster. If $V = 0$ then the index ν (33) follows already from the scaling (29) (there is no upper bound in equation (32) if $V = 0$).
2. The lower bound (32) is a result of the integration over \mathbf{x} in equation (30). The upper bound follows from the bound (26)

$$\begin{aligned}
 \int d\mathbf{x} \langle K_\tau^{(A,V)}(\mathbf{x}, \mathbf{x}) \rangle & \leq (2\pi\tau\sigma^2)^{-\frac{D}{2}} \int d\mathbf{x} E \left[\exp\left(-\frac{1}{\hbar} \int_0^\tau V(\mathbf{x} + \sqrt{s}\sigma\mathbf{a}_s) ds\right) \right] \\
 & = (2\pi\tau\sigma^2)^{-\frac{D}{2}} \int d\mathbf{x} \exp\left(-\frac{\tau}{\hbar}V(\mathbf{x})\right).
 \end{aligned}
 \tag{34}$$

The upper bound (34) follows from that for $\int d\mathbf{x} \langle K_\tau^{(A,V)}(\mathbf{x}, \mathbf{x}) \rangle$ (Kato inequality) and has been derived earlier in [13] as a consequence of the Golden–Thompson inequality.

3. The lower bound of corollary 4 is suggesting that a growing random vector field has a similar effect as a growing scalar potential leading to localized states. For the harmonic oscillator (with the oscillation frequency ω) $\text{Tr}(\exp(-\tau H)) = (\sinh(\frac{\omega\tau}{2}))^{-1}$. Hence, an increase of the index ν in equation (33) agrees with the exact formula (the index ν has also been discussed in [34]). However, the exponential decrease in the lower bound on the lhs of equation (32) does not reflect the exact large time behaviour of the trace of the heat kernel of the Hamiltonian with a scalar potential.
4. We can obtain the general formula for $\langle K_\tau \rangle$ from the transverse case transforming a general potential to the transverse one and subsequently calculating the average over the gauge function χ as in equation (18). The behaviour for large distances would not change substantially in theorems 2 and 3 if we worked with an arbitrary gauge. We have assumed the transverse gauge in order to avoid difficulties with differentiability of the potentials. Let us explain the problem using as an example the square of the first term on the rhs of equation (21):

$$\int_0^\tau ds \int_0^\tau ds' E[(\text{div } \mathbf{A}'(\mathbf{q}_s) \text{ div } \mathbf{A}'(\mathbf{q}_{s'}))] = \int_0^\tau ds \int_0^\tau ds' E[\partial_j \partial'_k G_{jk}(\mathbf{q}_s, \mathbf{q}_{s'})]. \quad (35)$$

If the second-order derivatives of G are bounded, then the term (35) is bounded by $c\tau^2$. However, in equation (23) (without the projection on the transverse part) the second-order derivative behaves as $|\mathbf{x}|^{2\gamma-2}$, which is singular for $\gamma < 1$. Then, the large-distance behaviour will be $\tau^2 |\mathbf{x}|^{2\gamma-2} = \tau^{1+\gamma} |\tau^{-\frac{1}{2}} \mathbf{x}|^{2\gamma-2}$ in agreement with the scaling formula (29).

We consider now the Green functions \mathcal{G}_m defined as solutions of the equation

$$\left(-\mathcal{A} + \frac{1}{2}m^2\right) \mathcal{G}_m = \delta. \quad (36)$$

The Green function can also be defined as the kernel of the inverse operator in the Hilbert space of square integrable functions $L^2(d\mathbf{x})$ [35]. Then,

$$\left(-\mathcal{A} + \frac{1}{2}m^2\right)^{-1} = \int_0^\infty d\tau \exp\left(-\frac{1}{2}m^2\tau\right) \exp(\tau\mathcal{A}). \quad (37)$$

By means of an integration over τ of the diamagnetic inequality (26) we obtain an upper bound for the Green function in a magnetic field in terms of the Green function without the magnetic field

$$\langle \mathcal{G}_m^{(A,V)}(\mathbf{x}', \mathbf{x}) \rangle \leq \mathcal{G}_m^{(0,V)}(\mathbf{x}', \mathbf{x}).$$

Under the assumptions of theorem 3 we obtain the lower bound

$$\langle \mathcal{G}_m^{(A,V)}(\mathbf{x}, \mathbf{x}') \rangle \geq C |\mathbf{x} - \mathbf{x}'|^{-D+2} \exp\left(-\frac{a}{\hbar^2} |\mathbf{x} - \mathbf{x}'|^2 - \left(\frac{b}{\hbar} + m\right) |\mathbf{x} - \mathbf{x}'|\right). \quad (38)$$

The lower bound for the random magnetic field with the covariance (23) and the scalar potential V (27) follows from equation (28) by an integration over τ . A detailed estimate of the behaviour of such integrals as a function of $|\mathbf{x} - \mathbf{x}'|$ is complicated. Without detailed estimates we can obtain an exponential decay in $|\mathbf{x} - \mathbf{x}'|$ of the lower bound (38) for $\mathcal{G}_m(\mathbf{x}, \mathbf{x}')$ as follows from the first τ -independent term in the exponential on the rhs of equation (28). It is not clear whether this exponential decay comes solely from the unprecise lower bound or if it is an intrinsic property of growing vector potentials.

The Green function is gauge dependent as follows from equation (11). We define the diagonal

$$\mathcal{G}_m^{(A,V)}(\mathbf{x}, \mathbf{x}) - \mathcal{G}_m^{(0,0)}(\mathbf{x}, \mathbf{x}) = \int_0^\infty d\tau \exp\left(-\frac{1}{2}m^2\tau\right) (K_\tau^{(A,V)}(\mathbf{x}, \mathbf{x}) - K_\tau^{(0,0)}(\mathbf{x}, \mathbf{x})),$$

which is gauge independent (see [36] for some estimates close to the diagonal). From equation (7)

$$\begin{aligned} |\mathcal{G}_m^{(A,V)}(\mathbf{x}, \mathbf{x}) - \mathcal{G}_m^{(0,0)}(\mathbf{x}, \mathbf{x})| &\leq \int_0^\infty d\tau \exp\left(-\frac{1}{2}m^2\tau\right) (2\pi\tau)^{-\frac{D}{2}} \\ &\times E \left[\left| \int_0^\tau \mathbf{A}(\mathbf{q}) \circ d\mathbf{q} \right| + \int_0^\tau ds V(\mathbf{q}_s) \right]. \end{aligned} \tag{39}$$

Hence, if V is continuous and \mathbf{A} is Hölder continuous with any index $\gamma > 0$ (as in equation (23)) then the diagonal (39) is finite if $D \leq 3$. This is so because the τ -integrand on the rhs of equation (39) for a small τ is bounded by $\tau^{-\frac{D}{2} + \frac{1+\gamma}{2}}$ and for a large τ it decays exponentially. In fact, we have $\int A(\mathbf{q}) d\mathbf{q} = \int (\mathbf{A}(\mathbf{q}) - \mathbf{A}(\mathbf{x})) d\mathbf{q}$ (because \mathbf{q} is a closed path) and for an estimate of the rhs we can apply equations (43)–(46) (see similar estimates in [37]).

Applying the inequality $|\langle F \rangle| \leq \langle |F| \rangle$, the Schwartz inequality and an estimate of $\langle E[\left(\int_0^\tau \mathbf{A} d\mathbf{q}\right)^2] \rangle$ in equation (41) we obtain for $D \leq 3$ and $V \geq 0$

$$|\mathcal{G}_m^{(0,0)}(\mathbf{x}, \mathbf{x}) - \mathcal{G}_m^{(A,V)}(\mathbf{x}, \mathbf{x})| \leq C_1(m)|\mathbf{x}|^{2\alpha} + C_2(m), \tag{40}$$

where $2\alpha = \max(\gamma, 2\beta)$ and $C_j(m) > 0$.

4. The estimates

We discuss now the estimates leading to the results of theorems 2 and 3. The upper bound in equations (25), (26) is an elementary consequence of formula (19) (or the Kato inequality of proposition 1). For the lower bound we estimate the expectation value on the rhs of equation (24). An explicit calculation of the average over the electromagnetic field gives

$$\begin{aligned} E \left[\left\langle \left(\int_0^\tau \mathbf{A}(\mathbf{q}(s)) d\mathbf{q}_s \right)^2 \right\rangle \right] &= \int_0^\tau \frac{ds}{\tau} \int_0^\tau \frac{ds'}{\tau} (\mathbf{x} - \mathbf{x}') E[G(\mathbf{q}(s), \mathbf{q}(s'))](\mathbf{x} - \mathbf{x}') \\ &+ \tau\sigma^2 \int_0^\tau \int_0^\tau \frac{ds}{\tau} \frac{ds'}{\tau} E \left[\mathbf{b} \left(\frac{s}{\tau - s} \right) G(\mathbf{q}(s), \mathbf{q}(s')) \mathbf{b} \left(\frac{s'}{\tau - s'} \right) \right] \\ &+ \tau\sigma^2 \int_0^\tau d \left(\frac{s}{\tau - s} \right) \left(1 - \frac{s}{\tau} \right)^2 \sum_j E[G_{jj}(\mathbf{q}(s), \mathbf{q}(s))] \\ &- 2\sigma\sqrt{\tau} \int_0^\tau \frac{ds}{\tau} \int_0^s \frac{ds'}{\tau} E \left[\mathbf{b} \left(\frac{s}{\tau - s} \right) G(\mathbf{q}(s), \mathbf{q}(s')) \right] (\mathbf{x} - \mathbf{x}') \\ &- 2\sigma\sqrt{\tau} \int_0^\tau \frac{ds}{\tau} \int_0^s \frac{ds'}{\tau} E \left[\mathbf{b} \left(\frac{s'}{\tau - s'} \right) G(\mathbf{q}(s), \mathbf{q}(s')) \right] (\mathbf{x} - \mathbf{x}') \\ &+ \sqrt{\tau}\sigma \int_0^\tau \frac{ds}{\tau} E \left[\left(\mathbf{x}' - \mathbf{x} - \sqrt{\tau}\sigma \mathbf{b} \left(\frac{s}{\tau - s} \right) \right) \right. \\ &\times \left. \int_0^s d\mathbf{b} \left(\frac{s'}{\tau - s'} \right) G(\mathbf{q}(s), \mathbf{q}(s')) \right] \left(1 - \frac{s'}{\tau} \right). \end{aligned} \tag{41}$$

Note that from the possible nine terms in equation (41) there remained only six because

$$E \left[\int_0^\tau \mathbf{u}_s d\mathbf{b}(s) \right] = 0, \tag{42}$$

if \mathbf{u}_s depends on $\mathbf{b}(s')$ with $s' \leq s$ (non-anticipating integrals [29]).

In order to estimate the lower bound in equation (24) we need some estimates on the Ito integrals. We have [29]

$$E \left[\left(\int \mathbf{f} \, d\mathbf{b}_s \right)^2 \right] = \int E[\mathbf{f}^2] \, ds. \tag{43}$$

Let

$$F(s) = \int_0^s \mathbf{f}(s', \mathbf{b}(s')) \, d\mathbf{b}(s'). \tag{44}$$

Then from the Schwartz inequality

$$\left| E \left[\int_0^\tau ds \, h(s, \mathbf{b}(s)) F(s) \right] \right|^2 \leq \int_0^\tau ds \, E[h(s)^2] \int_0^\tau ds \, E[F(s)^2], \tag{45}$$

and for random \mathbf{f} and h

$$\left| E \left[\left\langle \int_0^\tau ds \, h(s, \mathbf{b}(s)) F(s) \right\rangle \right] \right|^2 \leq \int_0^\tau ds \, \langle E[h(s)^2] \rangle \left\langle \int_0^\tau ds \, E[F(s)^2] \right\rangle, \tag{46}$$

where from equation (43)

$$E[F(s)^2] = \int_0^s \mathbf{f}(s', \mathbf{b}(s'))^2 \, ds'. \tag{47}$$

For the lower bound in equation (24) we bound each of the six terms in equation (41) from above. Then, we obtain the lower bound for the heat kernel inserting the upper bound with the minus sign for each term in the exponential in equation (24).

Proof of theorem 2. The upper bound in theorem 2 is a direct consequence of the Kato inequality (proposition 1). For the lower bound (25) it is sufficient to insert the upper bound for each term in equation (41). The terms without the stochastic integrals in equation (41) can be estimated by means of the Schwartz inequality using the boundedness of V and G whereas for the term with the stochastic integral we apply inequalities (45) and (43). \square

Proof of theorem 3. We could obtain the lower bound (28) estimating equation (41). In particular, we could apply inequality (45) directly to the last term in equation (41) (then h and \mathbf{f} in equations (43)–(45) do not depend on the electromagnetic field). However, it is instructive to return to equation (21) in order to estimate the product of the terms in the square of $\int \mathbf{A} \, d\mathbf{q}$ directly. Then, h and f depend linearly on the electromagnetic field.

We do not estimate all the six terms which come from the square of $\int \mathbf{A} \, d\mathbf{q}$ in equation (20) ($\text{div} \mathbf{A} = 0$) but concentrate on three typical terms. The remaining three terms can be estimated in a similar way. First, let us consider the last term on the rhs of equation (20) which comes from the square of the stochastic integral (20) (it corresponds to the third term on the rhs of equation (41))

$$\begin{aligned} \int_0^\tau ds \, E[(\mathbf{A}(\mathbf{q}(s))\mathbf{A}(\mathbf{q}(s)))] &= \int_0^\tau ds \sum_j E[G_{jj}(\mathbf{q}(s), \mathbf{q}(s))] \\ &\leq 4(D - 1)2^{4\gamma} \tau (|\mathbf{x}|^{2\gamma} + |\mathbf{x}'|^{2\gamma}) + 2^{2\gamma} \sigma^\gamma \tau^{1+\gamma} \int_0^1 E[|\mathbf{a}(s)|^{2\gamma}]. \end{aligned} \tag{48}$$

In equation (48) the Hölder inequality

$$|\mathbf{a} + \mathbf{b}|^{2\gamma} \leq 2^{2\gamma-1} (|\mathbf{a}|^{2\gamma} + |\mathbf{b}|^{2\gamma}) \tag{49}$$

has been applied to the covariance G of equation (23) with $\mathbf{q}(s)$ defined in equation (8). The square of the last term of equation (21) can be estimated as follows:

$$\begin{aligned} & \int_0^\tau \frac{ds}{\tau} \int_0^s \frac{ds'}{\tau} E[\langle \mathbf{A}(\mathbf{q}(s))(\mathbf{x}' - \mathbf{x}) \mathbf{A}(\mathbf{q}(s'))(\mathbf{x}' - \mathbf{x}) \rangle] \\ &= \int_0^\tau \frac{ds}{\tau} \int_0^s \frac{ds'}{\tau} E[(\mathbf{x}' - \mathbf{x}) G(\mathbf{q}(s), \mathbf{q}(s'))(\mathbf{x}' - \mathbf{x})] \\ &\leq a_1(|\mathbf{x}|^{2\gamma} + |\mathbf{x}'|^{2\gamma})(\mathbf{x} - \mathbf{x}')^2 + a_6|\mathbf{x} - \mathbf{x}'|^2 \sigma^\gamma \tau^\gamma, \end{aligned} \tag{50}$$

where for an estimate of $G(\mathbf{q}(s), \mathbf{q}(s'))$ the same set of inequalities has been applied as in equation (48).

Next, let us consider a term in the square of $\int \mathbf{A} d\mathbf{q}$ which is of the form of the expression appearing on the lhs of equation (46) (cf equations (20), (21) to see where this term comes from)

$$I = \sigma^2 \tau \left\langle E \left[\int_0^\tau \frac{ds}{\tau} \mathbf{A}(\mathbf{q}(s)) \mathbf{b} \left(\frac{s}{\tau - s} \right) \int_0^s \mathbf{A}(\mathbf{q}(s')) \left(1 - \frac{s'}{\tau} \right) d\mathbf{b} \left(\frac{s'}{\tau - s'} \right) \right] \right\rangle. \tag{51}$$

Now,

$$h(s, \mathbf{b}(s)) = \mathbf{A}(\mathbf{q}(s)) \mathbf{b} \left(\frac{s}{\tau - s} \right) \tag{52}$$

and

$$\mathbf{f} \left(s', \mathbf{b} \left(\frac{s'}{\tau - s'} \right) \right) = \mathbf{A}(\mathbf{q}(s')) \left(1 - \frac{s'}{\tau} \right). \tag{53}$$

Therefore in equation (46)

$$\langle E[h^2] \rangle = E \left[\mathbf{b} \left(\frac{s}{\tau - s} \right) G(\mathbf{q}(s), \mathbf{q}(s)) \mathbf{b} \left(\frac{s}{\tau - s} \right) \right] \tag{54}$$

and

$$\begin{aligned} & \int_0^\tau ds \int_0^s ds' \left\langle E \left[\mathbf{f}^2 \left(s', \mathbf{b} \left(\frac{s'}{\tau - s'} \right) \right) \right] \right\rangle \\ &= \int_0^\tau ds \int_0^s ds' \left(1 - \frac{s'}{\tau} \right)^2 \sum_j E[G_{jj}(\mathbf{q}(s'), \mathbf{q}(s'))]. \end{aligned} \tag{55}$$

Hence, from equations (45) and (46) and inequality (49) we obtain an estimate on I ,

$$|I| \leq a_3 \tau (|\mathbf{x}|^{2\gamma} + |\mathbf{x}'|^{2\gamma}) + a_5 \tau^{1+\gamma}, \tag{56}$$

On the basis of equation (41) and the estimates (48)–(56) it is clear how the lower bounds in equations (25) and (28) come out (the estimate in the lower bound (28) on the potential V (27) is a simple consequence of inequalities (49) and (24)). \square

5. Discussion and outlook

First of all, let us point out that a deterministic linearly rising vector potential really can lead to exponentially decaying Green functions.

The heat kernel of the Hamiltonian with a constant magnetic field \mathbf{B} in $D = 3$ satisfies the inequality [19, 14] (transverse gauge, the z axis is in the direction of \mathbf{B} and $\mathbf{x} = (x, y, z)$)

$$\begin{aligned}
 |K_\tau(x, y, z; x', y', z')| &= (2\pi\tau)^{-\frac{1}{2}} B \left(4\pi \sinh B \frac{\tau}{2}\right)^{-1} \\
 &\times \left| \exp\left(-\frac{1}{2\tau} B^2(z-z')^2 - \frac{B^3}{4} \coth\left(B \frac{\tau}{2}\right) ((x-x')^2 \right. \right. \\
 &\quad \left. \left. + (y-y')^2) + \frac{iB}{2}(xy' - x'y)\right)\right| \leq (2\pi\tau)^{-\frac{1}{2}} B \left(4\pi \sinh\left(B \frac{\tau}{2}\right)\right)^{-1} \\
 &\times \exp\left(-\frac{1}{2\tau} B^2(z-z')^2 - \frac{B^3}{4} ((x-x')^2 + (y-y')^2)\right),
 \end{aligned}$$

as

$$\coth\left(\frac{B\tau}{2}\right) \geq 1.$$

By an integration over τ (37) we obtain an upper bound on the Green function

$$\begin{aligned}
 |\mathcal{G}_m(x, y, z; x', y', z')| \\
 \leq C \exp\left(-\frac{B^3}{4} ((x-x')^2 + (y-y')^2) - B^{\frac{3}{2}} |z-z'| - m|\mathbf{x}-\mathbf{x}'|\right) \tag{57}
 \end{aligned}$$

(in the τ -integral the inequality $\sinh \frac{B\tau}{2} \leq \frac{1}{2} \exp\left(\frac{B\tau}{2}\right)$ has been applied).

The decay of the Green function (57) supports a heuristic argument that the term \mathbf{A}^2 in the Hamiltonian (2) is acting like a potential (see [41] and the precise results of [20, 21]). Note that the diagonal

$$K_\tau(x, y, z; x, y, z) = F(\tau) \tag{58}$$

being independent of any spatial coordinate, is not an integrable function in any of the components of \mathbf{x} .

From the formula for the heat kernel in terms of eigenfunctions and eigenvalues one can study their dependence on the random magnetic field. For this purpose we would need an upper bound for the heat kernel which is stronger than the diamagnetic inequality (26) of theorem 3. We are unable to derive such estimates in general. In order to study the localization effects of a random magnetic field we investigate a particular model. Let us assume that the magnetic field depends only on coordinates (x, y) of the XY plane. Then, we can choose $\mathbf{A} = (0, 0, A_3(x, y))$. In such a case the Hamiltonian $-\hbar\mathcal{A}$ reads

$$\hbar\mathcal{A} = -\frac{1}{2m}(p + A_3)^2 - V_3(z) + \frac{\hbar^2}{2m}\Delta_{xy} - V_2(x, y), \tag{59}$$

where Δ_{xy} is the two-dimensional Laplacian. In equation (59) we added a potential $V_3(z)$ ensuring a localization in z . We investigate the conditions on V_2 which imply a finite trace of $\langle \exp \tau \mathcal{A} \rangle$. We apply the Golden–Thompson inequality [38, 39] (for a precise formulation and the assumptions see [40])

$$\text{Tr}(\exp(\tau\mathcal{A})) \leq \text{Tr}\left(\exp\left(\frac{\tau}{2}\mathcal{B}\right)\exp(\tau\mathcal{C})\exp\left(\frac{\tau}{2}\mathcal{B}\right)\right) \tag{60}$$

if $\mathcal{A} = \mathcal{B} + \mathcal{C}$. The rhs of inequality (32) is a consequence of the Golden–Thompson inequality. Now, we choose

$$\hbar\mathcal{C} = -\frac{1}{4m}(p + A_3)^2 - V_3(z) - V_2(x, y) \tag{61}$$

and

$$\hbar\mathcal{B} = -\frac{1}{4m}(p + A_3)^2 + \frac{\hbar^2}{2m}\Delta_{xy}. \tag{62}$$

We have

$$\begin{aligned} \exp(\tau\mathcal{B})(\mathbf{x}; \mathbf{x}') &= (2\pi\sigma^2\tau)^{-1} \exp\left(-\frac{1}{2\tau\sigma^2}(x-x')^2 - \frac{1}{2\tau\sigma^2}(y-y')^2\right) \\ &\times \int dp \exp\left(\frac{i}{\hbar}p(z'-z)\right) E\left[\exp\left(-\frac{1}{4\hbar m} \int_0^\tau (p+A_3(\mathbf{q}(s)))^2 ds\right)\right]. \end{aligned} \quad (63)$$

Here $\mathbf{q} = (q_1, q_2)$ is two dimensional (the components defined in equation (8))

$$\begin{aligned} \exp(\tau\mathcal{C})(x, y, z; x', y', z') &= (\pi\sigma^2\tau)^{-\frac{1}{2}} \exp\left(-\frac{1}{\tau\sigma^2}(z-z')^2\right) \delta(x-x')\delta(y-y') \\ &\times \exp\left(-\frac{\tau}{\hbar}V_2(x, y)\right) \exp\left(\frac{i}{\hbar}(z'-z)A_3(x, y)\right) E\left[\exp\left(-\frac{1}{\hbar} \int_0^\tau V_3(q_3(s)) ds\right)\right], \end{aligned} \quad (64)$$

where

$$q_3(s) = z + (z' - z)\frac{s}{\tau} + \sqrt{\frac{\tau}{2}}\sigma a_3\left(\frac{s}{\tau}\right).$$

From the Golden–Thompson inequality

$$\langle \text{Tr}(\exp(\tau\mathcal{A})) \rangle \leq \int d\mathbf{x} \int d\mathbf{x}' \langle \exp(\tau\mathcal{B})(\mathbf{x}, \mathbf{x}') \exp(\tau\mathcal{C})(\mathbf{x}', \mathbf{x}) \rangle. \quad (65)$$

The expectation value over the magnetic field on the rhs of equation (65) can explicitly be calculated. We perform the calculations in a special case when the potential V_3 is quadratic,

$$V_3(z) = m\omega^2 z^2. \quad (66)$$

In such a case the expectation value over q_3 gives the heat kernel of the harmonic oscillator. After a calculation of integrals over z and z' we obtain

$$\begin{aligned} \langle \text{Tr}(\exp(\tau\mathcal{A})) \rangle &\leq (2\pi\tau\sigma^2)^{-\frac{1}{2}} (\sinh(\omega\tau))^{-1} \int dx dy \exp\left(-\frac{\tau}{\hbar}V_2(x, y)\right) \\ &\times \int dp \left\langle \exp\left(-\frac{1}{2m\omega} \sinh(\omega\tau)(\cosh(\omega\tau) + 1)^{-1}(p + A_3(x, y))^2\right) \right. \\ &\times E\left[\exp\left(-\frac{1}{4\hbar m} \int_0^\tau (p + A_3(\mathbf{q}(s)))^2 ds\right)\right] \Bigg\rangle \\ &\leq (2\pi\tau\sigma^2)^{-\frac{1}{2}} (\sinh(\omega\tau))^{-1} \int dx dy \exp\left(-\frac{\tau}{\hbar}V_2(x, y)\right) \\ &\times \int dp \int_0^\tau \frac{ds}{\tau} \left\langle E\left[\exp\left(-\frac{1}{2m\hbar\omega} \sinh(\omega\tau)(\cosh(\omega\tau) + 1)^{-1} \right. \right. \right. \\ &\times \left. \left. \left. (p + A_3(x, y))^2 - \frac{\tau}{4\hbar m} (p + A_3(\mathbf{q}(s)))^2\right)\right] \right\rangle. \end{aligned} \quad (67)$$

The expectation value over the magnetic field on the rhs of equation (67) can be calculated with the result (for the covariance (23))

$$\begin{aligned} \langle \text{Tr}(\exp(\tau\mathcal{A})) \rangle &\leq C_1(2\pi\tau\sigma^2)^{-1} (\sinh(\omega\tau))^{-1} \int dx dy \exp\left(-\frac{\tau}{\hbar}V_2((x, y))\right) \\ &\times \int_0^\tau \frac{ds}{\tau} E\left[(G((x, y), (x, y)) + G(\mathbf{q}_s, \mathbf{q}_s))^{-\frac{1}{2}}\right] \\ &\leq C_2(2\pi\tau\sigma^2)^{-1} (\sinh(\omega\tau))^{-1} \int dx dy \exp(-\tau V_2(x, y))(x^2 + y^2)^{-\frac{\nu}{2}}. \end{aligned} \quad (68)$$

Equation (68) shows that the growing random electromagnetic field improves localization. As an example we consider

$$V_2 = |x|^\alpha |y|^\alpha \tag{69}$$

(the case $\alpha = 2$ has been discussed by Simon [42]). The classical criterion for a discrete spectrum (equation (68) with $\gamma = 0$) is not satisfied (the region in the phase space with the classical energy less than E has an infinite volume, see [14, 42]). However, any $\gamma > 0$ (random vector field with a growing covariance) leads to a finite trace. Note that the results of [20, 21] concerning the discrete spectrum do not apply directly to the vector potential (23) and the scalar potential (69) because the covariance of the magnetic field \mathbf{B} is decaying as $|\mathbf{x}|^{2\gamma-2}$ ($\gamma < 1$). Hence, it is bounded in the mean.

In model (59) (with $V_3 = 0$) we can obtain some estimates on the off-diagonal of the heat kernel as well. Let $\tilde{K}_\tau(p; x, y, ; x', y')$ be the Fourier transform of $K_\tau(x, y, z; x', y', z')$ in $z' - z$; then

$$\begin{aligned} \tilde{K}_\tau(p; x, y, ; x', y') &= (2\pi\sigma^2\tau)^{-1} \exp\left(-\frac{1}{2\tau\sigma^2}(x-x')^2 - \frac{1}{2\tau\sigma^2}(y-y')^2\right) \\ &\quad \times E \left[\exp\left(-\frac{1}{\hbar} \int_0^\tau \left(\frac{1}{2m}(p + A_3(\mathbf{q}(s)))^2 + V_2(\mathbf{q}(s))\right) ds\right) \right] \\ &\leq (2\pi\sigma^2\tau)^{-1} \exp\left(-\frac{1}{2\tau\sigma^2}(x-x')^2 - \frac{1}{2\tau\sigma^2}(y-y')^2\right) \\ &\quad \times \int_0^\tau \frac{ds}{\tau} E \left[\exp\left(-\frac{\tau}{2\hbar m}(p + A_3(\mathbf{q}(s)))^2 - \frac{\tau}{\hbar} V_2(\mathbf{q}(s))\right) \right]. \end{aligned} \tag{70}$$

The expectation value over A_3 can be calculated exactly. Let us consider a simple case of a translation-invariant Gaussian field with $G(\mathbf{x}, \mathbf{x}') = G(\mathbf{x} - \mathbf{x}')$ (there is no scale invariance if $G(\mathbf{0})$ is finite). After a calculation of the expectation value on the rhs of equation (70) we obtain

$$\begin{aligned} \langle \tilde{K}_\tau(p; x, y, ; x', y') \rangle &\leq (2\pi\sigma^2\tau)^{-1} \exp\left(-\frac{1}{2\tau\sigma^2}(x-x')^2 - \frac{1}{2\tau\sigma^2}(y-y')^2\right) \\ &\quad \times \int_0^\tau \frac{ds}{\tau} E \left[\exp\left(-\frac{1}{2} p^2 \left(\frac{m\hbar}{\tau} + G(0)\right)^{-1} - \frac{\tau}{\hbar} V_2(\mathbf{q}(s))\right) \right] \left(1 + \frac{\tau}{m\hbar} G(0)\right)^{-\frac{1}{2}}. \end{aligned} \tag{71}$$

When $\tau \rightarrow \infty$ then the upper bound of equation (71) decreases as $\exp\left(-\frac{1}{2G(0)}p^2\right)$ for a large p . We could interpret such a decay of the Fourier transform of the heat kernel as a confirmation of the behaviour $\exp(-a_1\hbar^{-2}(z-z')^2)$ (which also has a Gaussian Fourier transform) in the lower bound (25) of theorem 2 (scale invariance of the vector potential has not been assumed there).

The decay of Green functions is important for correlation functions of the complex scalar fields interacting with an electromagnetic field in the Ginzburg–Landau model

$$\langle \phi^*(\mathbf{x})\phi(\mathbf{x}') \rangle = \langle \mathcal{G}_m^{(A,0)}(\mathbf{x}, \mathbf{x}') \rangle, \tag{72}$$

where $\mathcal{G}_m^{(A,0)}$ is defined in equation (37).

The lower and upper bounds on the higher-order correlations of the scalar fields can be studied by means of our methods as well. In such a case the integral $\int \mathbf{A} d\mathbf{q}$ must be extended to many paths joining the points \mathbf{x}_j as the arguments of the scalar fields ϕ . For example

$$\begin{aligned}
 \langle \phi^*(\mathbf{x})\phi^*(\mathbf{y})\phi(\mathbf{y}')\phi(\mathbf{x}') \rangle &= \langle \mathcal{G}^{(\mathbf{A},0)}(\mathbf{x}, \mathbf{x}')\mathcal{G}^{(\mathbf{A},0)}(\mathbf{y}, \mathbf{y}') \rangle + (\mathbf{x} \rightarrow \mathbf{y}) \\
 &= \int d\tau d\tau' (2\pi\tau\sigma^2)^{-\frac{D}{2}} (2\pi\tau'\sigma^2)^{-\frac{D}{2}} \exp\left(-\frac{1}{2\tau'\sigma^2}(\mathbf{y}-\mathbf{y}')^2 - \frac{1}{2\tau\sigma^2}(\mathbf{x}-\mathbf{x}')^2\right) \\
 &\quad \times \left\langle E \left[\exp\left(\frac{i}{\hbar} \int_0^{\tau'} \mathbf{A}(\mathbf{q}_{\mathbf{y}\mathbf{y}'} \circ d\mathbf{q}_{\mathbf{y}\mathbf{y}'}) + \frac{i}{\hbar} \int_0^{\tau} \mathbf{A}(\mathbf{q}_{\mathbf{x}\mathbf{x}'} \circ d\mathbf{q}_{\mathbf{x}\mathbf{x}'}) \right) \right] \right\rangle + (\mathbf{x} \rightarrow \mathbf{y}),
 \end{aligned}
 \tag{73}$$

where $(\mathbf{x} \rightarrow \mathbf{y})$ means the same expression but with exchanged arguments. We can calculate the expectation value over the electromagnetic field and derive upper and lower bounds for the correlation functions (73). The important question to be answered is whether the decay of correlations holds true for any two points tending to infinity in the multi-point correlation functions of the scalar fields. This problem needs further investigation.

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